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in which each of them stands to some third object which is the middle term. The pure general form of the syllogism, when its premises are absolutely asserted, is as follows: X is in the relation L to Y, Y is in the relation M to Z; therefore X is in the relation "L of M," compounded of L and M, to Z. In ordinary logic the actual composition of the relation is made by our consciousness of its *transitive* character. A relation is transitive when, being compounded with itself, it reproduces itself; that is, L is transitive when every L of L is L; for example "brother."

Thus De Morgan broke away from that paltry narrowness which asserts that our minds in pure thinking can use nothing but the relation of identity; from the Jevons sophism that thought cannot move because all thought is the substitution of identicals.

So we see that in logic, as in mathematics, we may develop a whole system of theorems about symbols which are to be used in a given manner; and then to make this whole system true of a desired relation or subject matter we have only to show that this relation or subject matter fulfills the few fundamental principles of the system.

De Morgan treated of convertible and inconvertible relatives, repeating relatives; non-repeating relatives, transitive and intransitive relatives, and inaugurated a general system.

To what tremendous estate this system has grown may be seen in the 649 pages of Ernst Schroeder's Treatise, "Algebra und Logik der Relative;" Leipzig, Teubner, 1895, on whose first page, as founder of the system, stands the name of Augustus De Morgan.

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ON THE SOLUTION OF THE QUADRATIC EQUATION.

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One of the most important applications of substitution groups occurs in the theory of the solution of algebraic equations. It seems desirable that a fairly complete discussion of the solution of the quadratic equation should precede the study of this application. It is hoped that this discussion will not be without interest in itself even if the facts with which we have to deal are well known. As we shall need a clear idea of the *domain of rationality* we shall first develop several elementary concepts which involve the notion of groups and naturally lead to the more general concept of the domain of rationality.

Let us first consider the totality of numbers (T_1) formed by all the positive integers,* each positive integer occurring once and only once. By adding any one of these to itself or to any other number in T_1 we obtain no new number. We may therefore say that T_1 forms a *group* (G_1) *with respect to addition.* It is evident that T_1 also forms a group (G_2) with respect to multiplication. Each of these two groups contains an indefinite number of subgroups.[*] To every subgroup of G_1 there corresponds a subgroup of G_2 which contains the same numbers but the converse is not true. For instance, all the positive integral powers of any prime number form a subgroup of G_2[†] but they do not form a subgroup of G_1.

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*We shall throughout confine our attention to the finite. Not only are the numbers to be regarded as finite but the number of times that a given operation is to be performed is also to be considered finite.

We shall not enter upon the discussion of the subgroups found in G_1 and G_2 but only call attention to a few of the most simple ones. All the even positive integers clearly form a subgroup of both of these groups. We may inquire what is the smallest subgroup that contains any given positive integer a. In G_2 this subgroup is a^α, $\alpha=1, 2, 3, \ldots\ldots$ while in G_1 this subgroup is βa, $\beta=1, 2, 3, \ldots\ldots$ Hence unity is a subgroup of G_2 but not of G_1. As G_1 contains no subgroup that involves unity we may say that it is generated by this element. G_2 is generated by all the prime numbers together with unity.

The totality of negative integers (T_2 also forms a group with respect to addition but it does not form a group with respect to multiplication. It is clear that the smallest group with respect to multiplication that contains T_2 must also contain T_1. That is, $T_1+T_2=T_3$ is a totality which forms a group with respect to multiplication. T_3 also forms a group with respect to each of the operations addition and subtraction. The group with respect to subtraction does not contain any subgroup involving either T_1 or T_2, that with respect to multiplication contains a subgroup that involves T_1 but none that involves T_2, while that with respect to addition contains a subgroup that includes T_1 and also one that includes T_2.

Among the numbers which are now in common use those included in T_1 are perhaps of special importance as is also indicated by the fact that they are frequently called the natural numbers. In considering the groups which certain totalities of numbers form with respect to given operations it is therefore of special importance to inquire into the smallest groups that contain T_1. We have already seen that with respect to subtraction this smallest group includes other numbers than those contained in T_1. Similarly we observe that with respect to division[‡] this smallest group includes an additional totality of numbers, viz: the fractions whose numerators and denominators are positive integers.

Instead of inquiring into the smallest totality of numbers that contains T_1 and forms a group with respect to a given operation we may also inquire into the smallest totality that contains T_1 and forms a group with respect to each of a given number of different operations. For instance, the smallest totality

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[*] The term subgroup is used, as usual, to represent a group that is contained in the larger group under consideration.

[†] For brevity we shall say that certain elements of a group form a subgroup instead of saying that these elements form a group if they are combined according to the same operation as the elements of the group.

[‡] As we have excluded the infinite we are not allowed to divide by 0. We may regard this as an impossible operation in the region to which we confine our observations.

(T_4) that contains T_1 and forms a group with respect to each of the four fundamental operations—addition, subtraction, multiplication, and division—is that which is formed by all the rational numbers. If we form the smallest totality that contains unity and forms a group with respect to each of these four operations we evidently arrive again at T_4. On this account the totality of all the rational numbers is generally called the domain unity. This is the simplest domain of rationality.*

It would be of interest to consider all the different types of subgroups of the groups formed by T_4 with respect to the given fundamental operations. We shall not enter into this field as the matter has probably been sufficiently developed for our present purposes. We would remark, however, that a careful study of these matters seems to us to be one of the simplest roads towards forming a clear notion of groups as well as of the domain of rationality.

The operations which we have thus far considered may be represented by the simple equations:

$$a+b=x, \quad a-b=x, \quad a\times b=x, \quad a\div b=x.$$

We have seen that x is always a rational number when both a and b are such numbers. In other words, we have seen that when we take a and b from the totality of numbers represented by T_4, x will also belong to this totality, but when we take a and b from one of the other totalities of numbers that have been considered—T_1, T_2, T_3—we cannot affirm that x belongs to the same totality in all the equations.

At an early stage in the development of mathematics it became necessary to solve equations of a higher than the first degree. One of the great difficulties which presented itself at this point was the fact that T_4 does not form a group with respect to the algebraic operations whose degree exceeds one. As long as these operations can be represented by a binomial equation of the form

$$x^n=a$$

n being a positive integer and a being a positive rational number, the difficulty was not much greater than that which had to be overcome at preceding stages, for the extension of the totality of numbers in such a way as to include irrational numbers does not seem an irrational adventure, even if the association of these numbers with matters of observation is not so direct and evident as it had been in the preceding cases.

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*The totality of rational functions with integral coefficients of a given number of determinate or indeterminate independent quantities (R, R', R'', ...) is called, after Kronecker, a domain of rationality. In other words, it is the smallest totality that contains these quantities and forms a group with respect to the four fundamental operations. In the domain unity we have clearly only one such quantity and this is determinate, viz: $R=1$. We shall soon consider a domain of one indeterminate quantity or parameter. Sometimes the given R's are defined as indeterminate parameters. According to this definition the domain unity contains no parameter. This domain is clearly generated by any rational finite number except 0. It may therefore be called the domain 2, 3, ... as well as the domain 1.

A much more serious difficulty presented itself when it was required to introduce the operation indicated by the general quadratic equations

$$x^2 + bx + c = 0. \qquad (A).$$

Even if we take b and c from the totality T_1 it often happens that we can not find any number among those that have been considered which comes near towards satisfying this equation. Hence the totality of real numbers (T_5) can clearly not form a group with respect to this operation. With respect to the four fundamental operations T_5 forms a group which contains T_4 as a subgroup.

It should however not be inferred that none of the numbers that have been considered form a group with respect to the operation (A). In order that a single number (α) may have this property it is necessary and sufficient that $x = \alpha$ satisfies the equation

$$x^2 + \alpha x + \alpha = 0.$$

Hence there are two numbers ($-\frac{1}{2}$ and 0) each of which forms a group with respect to the quadratic equation. 0 also forms a group with respect to the four fundamental operations but $-\frac{1}{2}$ does not have this property. It should also be observed that when $\alpha = -\frac{1}{2}$ it is not the only value of x that satisfies the given equation and that the term group has therefore to used in a somewhat restricted sense when we say that $-\frac{1}{2}$ forms a group with respect to the quadratic equation.

But, even if it was known that certain special numbers form a group with respect to the operation (A), and, what is more important, that x belongs to the totality of numbers T_5 for a large number of types of (A) yet the matter remained in an unsatisfactory state as long as no totality of numbers was known which includes T_5 and forms a group with respect to (A). The struggle for light on this point was a long one, reaching far into our century. We cannot enter into a history of this struggle. It must suffice to state that the triumph was largely due to the elegant geometrical representation of the complex number by means of points in the plane. This was not the first nor last assistance that algebra has received from his sister geometry. On the other hand, algebra has a very brilliant record of services rendered to his ambitious sister.

The importance of the adoption of the complex numbers (T) cannot be fully appreciated if we confine our attention to the quadratic equation. If the general algebraic operations of each of the following degrees had again required equally great extension in the number system this would soon have become exceedingly difficult and the progress in the solution of the algebraic equations could not have been so rapid. The great importance of the adoption of the totality of numbers T may therefore be said to be due to the fact that it forms a group with respect to the general algebraic operation indicated by the following equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n = 0 \qquad (B),$$

where a_0, a_1, a_2, ..., a_n belong to T and n belongs to T_1. In other words if the coefficients of (B) belong to the domain of the indeterminate parameter

$$R=\sqrt{a} \qquad (a \text{ belonging to } T_5)$$

x will also belong to this domain. Or again, B can be resolved into factors without using numbers that lie outside of T.

While we cannot fully appreciate the importance of the adoption of the complex numbers when we confine our attention to the quadratic equation, yet in this equation we see the source of T and with it we naturally associate the wonderful progress achieved by means of T. It is however not our object to enter upon the discussion of the important position which the quadratic equation occupies in the development of mathematics even if an idea of this position naturally increases the interest in this equation and hence also in its discussion as an algebraic operation.

To convey an idea of the importance of the concept domain of rationality we shall consider an application. Suppose that we have an equation of the form (B) and that its coefficients belong to a certain domain of rationality T' while none of its factors belong to this domain.* Suppose that we have any other equation whose coefficients also belong to T' and that these two equations have a common root. We can then say that the first equation is a factor of the second. In other words, we know that all the roots of the first equation are also roots of the second. The truth of this statement follows directly from the fact that the coefficients of the greatest common divisor of the two equations must belong to T'. This greatest common divisor must therefore be the first member of the first equation.

If we have any complex number

$$a+bi$$

the quadratic equation which contains this number and its conjugate for its roots is

$$x^2-2ax+a^2+b^2=0. \qquad (C).$$

The coefficients of this equation belong to T_5. Suppose now that we have any other equation that contains $a+bi$. From the proof just given it follows that it must also contain (C). The same remarks clearly apply to surd roots of the form

$$a\pm\sqrt{b}.$$

Hence we see that the given statement includes the theorems that if an equation with real coefficients contains the root $a+bi$ it also contains the root $a-bi$, and if an equation whose coefficients are rational contains a surd root of the form $a+\sqrt{b}$ it also contains $a-\sqrt{b}$ as a root.

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*In this case we say that the equation is irreducible in the domain T'.